

Some Characterizations of VNL Rings

Harpreet K.Grover¹ and Dinesh Khurana²

Department of Mathematics, Panjab University, Chandigarh-160014, India

harpreetgr@gmail.com, dkhurana@pu.ac.in

Abstract

A ring R is said to be VNL if for any $a \in R$, either a or $1-a$ is (von Neumann) regular. The class of VNL rings lies properly between the exchange rings and (von Neumann) regular rings. We characterize abelian VNL rings. We also characterize and classify arbitrary VNL rings without infinite set of orthogonal idempotents; and also the VNL rings having primitive idempotent e such that eRe is not a division ring. We prove that a semiperfect ring R is VNL if and only if for any right uni-modular row $(a_1, a_2) \in R^2$, one of the a_i is regular in R . Formal triangular matrix rings that are VNL, are also characterized. As a corollary it is shown that an upper triangular matrix ring $T_n(R)$ is VNL if and only if $n = 2$ or 3 and R is a division ring.

1. Introduction

As a common generalization of local and (von Neumann) regular rings, Contessa in [5] called a ring R VNL (von Neumann local) if for each $a \in R$, either a or $1-a$ is (von Neumann) regular. As every regular element a of a ring R is an exchange element (in the sense that there exists an idempotent $e \in aR$ such that $1-e \in (1-a)R$), VNL rings are exchange rings. But if R is a local ring with nonzero $J(R)$, then $R \times R$, which is an exchange ring, is not a VNL ring. For instance, if $a = (x, 1-x)$, where x is a nonzero element in $J(R)$, then neither a nor $1-a$ is regular. Although VNL rings have been studied in some detail (see [3], [4], [8] and [9]), their structure is not known even in commutative case. For instance, Osba, Henriksen and Alkam in ([8], page 2641) remark:

We are unable to characterize (commutative) VNL-rings abstractly in the sense of relating them to more familiar classes of rings...

¹The research of the first author is supported by CSIR, India and will form part of her Ph.D Thesis

²Corresponding author

The present paper is an effort towards this direction. We characterize abelian³ VNL rings. It is shown in Section 3 below that abelian VNL rings are precisely those exchange rings R in which, for any idempotent e , one of the two corner rings eRe and $(1 - e)R(1 - e)$ is regular. Let $M(R)$ denote the maximal regular ideal of a ring R as defined by Brown and McCoy in [1]. In ([3], Lemma 2.7) Chen and Tong showed that if R is an abelian VNL ring, then $R/M(R)$ is a local ring. We show that abelian VNL rings are precisely those rings R for which $R/M(R)$ is a local ring. But this characterization of abelian VNL rings is not valid for arbitrary rings (see Example 3.3 below). In Section 4, we characterize arbitrary VNL rings which do not have infinite set of orthogonal idempotents. As an exchange ring without infinite set of orthogonal idempotents is semiperfect (see [2]), this gives us characterization of semiperfect VNL rings. We prove that a semiperfect ring R is VNL if and only if for any right uni-modular row $(a_1, a_2) \in R^2$, one of the a_i is regular in R . We also characterize VNL rings R with a primitive idempotent e such that eRe is not a division ring or equivalently $J(eRe) \neq 0$ (if e is a primitive idempotent in an exchange ring R , then eRe is a local ring).

In Section 2, we give some examples of VNL rings and prove some basic properties of VNL rings. For instance, it is proved that if e is an idempotent in a VNL ring R , then either eRe or $(1 - e)R(1 - e)$ is a regular ring. We also show that this property does not characterize VNL rings. We also understand the regular elements of formal triangular matrix ring $\begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$ and with the help of this, we characterize formal triangular matrix rings, that are VNL. As a corollary, we prove that the upper triangular matrix ring $T_n(R)$ is VNL if and only if $n = 2$ or 3 and R is a division ring.

In [6] Nicholson defined a ring R to be NJ if every element of $R \setminus J(R)$ is regular. Clearly, an NJ ring R is VNL and Nicholson proved that in an NJ-ring, eRe is regular for every proper idempotent e of R . In Section 5, we prove that if R is a ring without a nontrivial central idempotent and $J(R) \neq 0$, then R is NJ if and only if R is VNL and $J(eRe) = 0$ for every proper idempotent e of R .

2. Examples and Basic Properties of VNL rings

The trivial examples of VNL rings, of course, are regular and local rings.

³A ring is called abelian if its all idempotents are central.

Here we give some non-trivial examples of VNL rings.

Examples 2.1

(1) Let $R = \{(q_1, q_2, \dots, q_n, z, z, z, \dots) : n \geq 1, q_i \in \mathbb{Q} \text{ and } z \in \mathbb{Z}_2\}$ where \mathbb{Z}_2 denotes the localization of \mathbb{Z} at the prime ideal (2). Then R is a VNL ring. An element $(q_1, q_2, \dots, q_n, z, z, z, \dots)$ is regular precisely when z is a unit in \mathbb{Z}_2 . It is easy to see that every non-zero ideal of R contains a non-zero idempotent implying that $J(R) = 0$. Thus VNL rings may not be semiregular.

(2) Nicholson in [6] studied the rings R with the property that every element outside $J(R)$ is regular. He called these rings NJ-rings. Clearly, every NJ ring is VNL. He characterized NJ rings by showing that the only NJ rings besides regular and local rings are of the form $\begin{pmatrix} D_1 & X \\ Y & D_2 \end{pmatrix}$, where D_1 and D_2 are division rings, X and Y respectively are (D_1, D_2) and (D_2, D_1) bi-modules with $XY = 0 = YX$. If D is a division ring, then from Nicholson's characterization of NJ rings, it is clear that the upper triangular matrix ring $T_2(D)$ is an NJ ring and hence VNL.

(3) It was observed in [8] that the ring \mathbb{Z}_n of integers mod n is VNL if and only if $(pq)^2$ does not divide n where p and q are distinct primes. This is clear from the fact that if $R \times S$ is a VNL ring, then either R or S is regular.

(4) For a commutative ring R , the formal power series ring $R[[x]]$ is VNL if and only if R is local (see [8]).

(5) If R is a regular ring, L is a local ring and ${}_R M_L$ is a bimodule then $\begin{pmatrix} R & M \\ 0 & L \end{pmatrix}$ is VNL. In fact, we show that every element of the type $\begin{pmatrix} r & m \\ 0 & l \end{pmatrix}$, where l is a unit in L , is regular. Since r is regular so there exists an $s \in R$ such that $rsr = r$, then as l is a unit in L , so

$$\begin{pmatrix} r & m \\ 0 & l \end{pmatrix} \begin{pmatrix} s & -sml^{-1} \\ 0 & l^{-1} \end{pmatrix} \begin{pmatrix} r & m \\ 0 & l \end{pmatrix} = \begin{pmatrix} r & m \\ 0 & l \end{pmatrix}.$$

We now prove some basic results about VNL rings.

Proposition 2.2. *Let R be a VNL ring then center of R is also a VNL ring.*

Proof. Let $x \in Z(R)$ be regular then there exists $y \in R$ such that $x = yxy$. Then it is easy to see that $z = yxy \in Z(R)$ and $x = xzx$ i.e x is regular

in $Z(R)$. So center of R is also VNL. \square

The following corollary is immediate from above result:

Corollary 2.3. *Let R be a VNL ring, then it is indecomposable as a ring if and only if its center is a local ring.*

If $R = S \times T$ is VNL, then it is clear that either S or T is a regular ring, because if s in S and t in T are non-regular elements then neither $r = (s, 1-t)$ nor $1-r = (1-s, t)$ is regular. Thus if e is a central idempotent in a VNL ring R , then either eRe or $(1-e)R(1-e)$ is regular. Interestingly, this also holds for non-central idempotents of VNL rings as shown below.

Lemma 2.4. *If R is a VNL ring then for every idempotent e of R , either eRe or $(1-e)R(1-e)$ is a regular ring.*

Proof. Let R be a VNL ring and $e \in R$ be an idempotent. Then

$$R \cong \begin{pmatrix} eRe & eR(1-e) \\ (1-e)Re & (1-e)R(1-e) \end{pmatrix}.$$

If $x \in eRe$ and $y \in (1-e)R(1-e)$ are two non-regular elements, then both $a = \begin{pmatrix} x & 0 \\ 0 & 1-y \end{pmatrix}$ and $1-a = \begin{pmatrix} 1-x & 0 \\ 0 & y \end{pmatrix}$ are also non-regular. \square

The following example shows that the necessary condition of Lemma 2.4 above does not characterize VNL rings.

Example 2.5. Let $R = \{(q_1, q_2, \dots, q_n, z, z, z, \dots) : n \geq 1, q_i \in \mathbb{Q}, z \in \mathbb{Z}\}$. It is clear that for idempotent e of R , either eRe or $(1-e)R(1-e)$ is regular. But R is not a VNL ring as \mathbb{Z} is a homomorphic image of R , which is not VNL.

The following result also clearly follows from Lemma 2.4.

Corollary 2.6. *For a ring R , the matrix ring $M_n(R)$, $n > 1$, is VNL if and only if R is regular.*

In [8], Osba, Henriksen and Alkam defined a commutative ring R to be SVNL if $\sum_{i=1}^n a_i R = R$ implies that one of the a_i 's is regular, and asked if every commutative VNL ring R is SVNL. This question was answered by Chen and Tong in [3], where they, in fact, proved that whenever $\sum_{i=1}^n a_i R = R$ in an abelian VNL ring, then one of the a_i 's is regular. We give below a

different proof of their result.

Corollary 2.7 (Chen and Tong [3], Theorem 2.8). *Let R be an abelian VNL ring. If $\sum_{i=1}^n a_i R = R$, then one of the a_i 's is regular.*

Proof. As R is an exchange ring and $\sum_{i=1}^n a_i R = R$, there exist an orthogonal set $\{e_1, \dots, e_n\}$ of idempotents such that $e_i \in a_i R$ and $e_1 + \dots + e_n = 1$ (see [7], Proposition 1.11). Now for each i ,

$$e_i \in a_i R \implies a_i R + (1 - e_i)R = R \implies e_i a_i R = e_i R.$$

Thus $e_i a_i$ is regular for every i . By Lemma 2.4, either $e_i R = e_i R e_i$ or $(1 - e_i)R = (1 - e_i)R(1 - e_i)$ is regular for each i . If $e_i R e_i$ is regular for each i , then R , being a direct product of $e_i R e_i$, is regular. Also if $(1 - e_i)R(1 - e_i)$ is regular for some i , then, as $e_i a_i$ is already regular, $a_i = e_i a_i + (1 - e_i)a_i$ is regular. \square

We now characterize the regular elements of formal triangular matrix rings. The characterization turned out to be very useful in the investigation of VNL rings.

Proposition 2.8. *Let ${}_R M_S$ be a bimodule. An element $\begin{pmatrix} a & m \\ 0 & b \end{pmatrix}$ of the formal triangular matrix ring $T = \begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$ is regular if and only if there exist idempotents $e \in R$ and $f \in S$ such that $aR = eR$, $Sb = Sf$ and $(1 - e)m(1 - f) = 0$.*

Proof. If $\begin{pmatrix} a & m \\ 0 & b \end{pmatrix}$ is regular in T , then for some $\begin{pmatrix} x & y \\ 0 & z \end{pmatrix}$ in T

$$\begin{pmatrix} a & m \\ 0 & b \end{pmatrix} \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \begin{pmatrix} a & m \\ 0 & b \end{pmatrix} = \begin{pmatrix} a & m \\ 0 & b \end{pmatrix}.$$

So $axa = a$, $bzb = b$, $axm + ayb + mzb = m$. If we take $e = ax$ and $f = zb$, then

$$aR = eR, \quad Sb = Sf \text{ and } (1 - e)m(1 - f) = 0.$$

Conversely, let $aR = eR$, $Sb = Sf$ and $(1 - e)m(1 - f) = 0$ for some idempotents $e \in R$ and $f \in S$. Then $m = em + mf - emf$, $ar = e$ and $sb = f$ for some $r \in R$ and $s \in S$. Thus

$$\begin{pmatrix} a & m \\ 0 & b \end{pmatrix} \begin{pmatrix} r & -rms \\ 0 & s \end{pmatrix} \begin{pmatrix} a & m \\ 0 & b \end{pmatrix} = \begin{pmatrix} a & m \\ 0 & b \end{pmatrix}. \quad \square$$

For the characterization of formal triangular matrix rings that are VNL, we need the following

Definition 2.9. We call a module ${}_R M$ *partial* if for any idempotent $e \in R$, either $eM = 0$ or $(1 - e)M = 0$.

Here are some examples of partial modules.

Examples 2.10. (1) Any module over a ring R with only trivial idempotents is partial. In particular, every vector space is a partial module.

(2) Any simple module over a commutative ring R is partial.

(3) If S is a ring with only trivial idempotents, $R = S \times S \times \dots \times S$ and $M = S \times 0 \times \dots \times 0$, then clearly ${}_R M$ is a partial module.

It follows from the following result that no non-zero module over a proper matrix ring is partial.

Proposition 2.11. *For any ring R , let $S = M_n(R)$ with $n \geq 2$, and ${}_S M$ be a non-zero module. Then for $0 \neq m \in M$, there exists an idempotent e in S such that $em \neq 0$ and $(1 - e)m \neq 0$. In particular, no non-zero module over S is partial.*

Proof. In S

$$1 = E_{11} + E_{22} + \dots + E_{nn} \Rightarrow m = E_{11}m + E_{22}m + \dots + E_{nn}m.$$

Since m is non-zero, $E_{ii}m \neq 0$ for some i . If for some $j \neq i$, $E_{jj}m \neq 0$, then either $(1 - E_{ii})m \neq 0$ or $(1 - E_{jj})m \neq 0$, because otherwise $m = E_{ii}m = E_{jj}m$ implying that $m = 0$. Now suppose that there is only one i such that $E_{ii}m \neq 0$. Pick any $j \neq i$ and consider $e = E_{jj} + E_{ji}$. Then e is an idempotent in S . Now $em = E_{jj}m + E_{ji}m = 0 + E_{ji}m = E_{ji}m$. If $E_{ji}m = 0$, then $E_{ij}E_{ji}m = 0$ implying that $E_{ii}m = 0$. Hence $em \neq 0$. Also

$$(1 - e)m = m - E_{jj}m - E_{ji}m = m - E_{ji}m.$$

If $m - E_{ji}m = 0$, then $E_{ji}m - E_{ji}^2m = E_{ji}m = 0$. But this, as seen above, implies that $m = 0$. \square

In the following result we characterize the formal triangular matrix rings that are VNL.

Theorem 2.12. *Let ${}_R M_S$ be a bimodule. Then the ring $T = \begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$ is VNL if and only if*

- (1) *One of R and S is regular and the other is VNL.*
- (2) *Either ${}_R M$ or M_S is a partial module.*
- (3) *For any non-regular $r \in R$, $(1 - r)M = M$ and for any non-regular element $s \in S$, $M(1 - s) = M$.*

Proof. Suppose that T is VNL. Then, by Lemma 2.4 and the fact that every factor ring of a VNL ring is VNL, one of the R and S is regular and the other is VNL. Now suppose that ${}_R M$ is not a partial module. So there exist an idempotent $e \in R$ such that

$$eM \neq 0 \text{ and } (1 - e)M \neq 0. \quad (A)$$

Let f be an idempotent in S , then if we take idempotent $E = \begin{pmatrix} e & 0 \\ 0 & f \end{pmatrix} \in T$, by Lemma 2.4, either ETE or $(1 - E)T(1 - E)$ is regular. So

$$\text{either } eMf = 0 \text{ or } (1 - e)M(1 - f) = 0. \quad (B)$$

Similarly, if we take the idempotent $\begin{pmatrix} 1 - e & 0 \\ 0 & f \end{pmatrix} \in T$, we get

$$\text{either } (1 - e)Mf = 0 \text{ or } eM(1 - f) = 0. \quad (C)$$

From (A), (B) and (C), it is clear that either $Mf = 0$ or $M(1 - f) = 0$. Thus M_S is a partial module. Now suppose that r is a non-regular element in R . Then as $\begin{pmatrix} r & m \\ 0 & 1 \end{pmatrix}$ is not regular for any $m \in M$, the element $\begin{pmatrix} 1 - r & m \\ 0 & 0 \end{pmatrix}$ is regular for every $m \in M$. Now if $(1 - r)R = eR$, then by Proposition 2.8, $(1 - e)M = 0$. So $M = eM = (1 - r)M$. Similarly if s is a non-regular element of S , then $M(1 - s) = M$.

For converse, we may assume without loss of generality that R is VNL and S is regular. Let $x = \begin{pmatrix} r & m \\ 0 & s \end{pmatrix} \in T$. If r is regular in R but $1 - r$ is not regular, then we show that x is regular in T . As $1 - r$ is not regular, by condition (3), $rM = M$. Thus if $rR = eR$ for some idempotent $e \in R$, then $eM = M$ and so $(1 - e)M = 0$. This, by Proposition 2.8, implies that

x is regular. Now suppose that both r and $1 - r$ are regular in R . Suppose ${}_R M$ is a partial module. Now if $rR = eR$ for some idempotent $e \in R$, then either $eM = 0$ or $(1 - e)M = 0$. If $(1 - e)M = 0$, then, by Proposition 2.8, x is regular. Now if $eM = 0$, then $rM = 0$. So $(1 - r)M = M$ and if $(1 - r)R = fR$ for some idempotent f of R , then $fM = M$ implying that $(1 - f)M = 0$. Thus, by Proposition 2.8, $1 - x$ is regular. So T is VNL. Lastly if M_S is partial, we can similarly prove that T is VNL. \square

We now give various applications of Theorem 2.12. In [4], it was proved that if D is a division ring, then $T_2(D)$ and $T_3(D)$ are VNL. The following characterization shows that these are the only upper triangular matrix rings that are VNL.

Corollary 2.13. *The upper triangular matrix ring $T_n(R)$ is VNL if and only if $n = 2$ or 3 and R is a division ring.*

Proof. Let $n \geq 4$ and $e = E_{11} + E_{22}$. Then e is an idempotent in $T_n(R)$ and $eT_n(R)e \cong T_2(R)$, $(1 - e)T_n(R)(1 - e) \cong T_{n-2}(R)$ are both not regular. So by Lemma 2.4, $T_n(R)$ is not regular if $n \geq 4$. Also it is clear by Proposition 2.8 that any element outside the Jacobson radical of $T_2(D)$ is regular implying that $T_2(D)$ is VNL. Now $T_3(D) = \begin{pmatrix} T_2(D) & M \\ 0 & D \end{pmatrix}$. Clearly, $T_3(D)$ satisfies the conditions (1) and (2) of Theorem 2.12. Also if $r \in T_2(D)$ is non-regular, then $1 - r$ is a unit and so $(1 - r)M = M$, implying that $T_3(D)$ also satisfies the condition (3) of Theorem 2.12 and is thus VNL.

Now suppose that $T_2(R)$ is VNL. By Lemma 2.4, R is regular. If e is any non-trivial idempotent in R , then neither ${}_R R$ nor R_R is partial. So, by Theorem 2.12, $T(R)$ is not VNL. Thus R is a regular ring without non-trivial idempotents and is thus a division ring. Lastly if $T_3(R)$ is VNL, then so is $T_2(R)$ being a homomorphic image of $T_3(R)$, implying again that R is a division ring. \square

Corollary 2.14. (1) *If R is a regular ring and S is a local ring, then for any bimodule ${}_R M_S$, the ring $\begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$ is a VNL ring.*

(2) *For any division ring D , the ring $\begin{pmatrix} M_2(D) & M \\ 0 & T_2(D) \end{pmatrix}$, where $M =$*

$\begin{pmatrix} 0 & D \\ 0 & D \end{pmatrix}$, is VNL.

Proof. The part (1) is immediate from Theorem 2.12. Also, as $M_{T_2(D)}$ is partial and for any non-regular $x \in T_2(D)$, $1 - x$ is a unit, $\begin{pmatrix} M_2(D) & M \\ 0 & T_2(D) \end{pmatrix}$ satisfies all the three conditions of Theorem 2.12 and is thus VNL. \square

Corollary 2.15. *Let R and S be simple artinian rings and ${}_R M_S$ be a bimodule. Then the ring $T = \begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$ is VNL if and only if either $M = 0$ or one of R and S is a division ring.*

Proof. The ‘if’ part follows from Corollary 2.14(1). Conversely, suppose T is VNL. If M is non-zero and neither R nor S is a division ring, then ${}_R M$ and M_S are not partial by Proposition 2.11. This, in view of Theorem 2.12 implies that T is not VNL. \square

The above proof, in fact, shows if ${}_{M_n(R)} M_{M_m(S)}$ is a bimodule, then the ring $\begin{pmatrix} M_n(R) & M \\ 0 & M_m(S) \end{pmatrix}$ is VNL only if either $M = 0$ or either $n = 1$ or $m = 1$.

Corollary 2.16. *Let R be a commutative ring and I be non-zero ideal of R . Then $T = \begin{pmatrix} R & I \\ 0 & R \end{pmatrix}$ is VNL if and only if $R = F \times S$ where F is a field and S is a regular ring and $I = F \times 0$.*

Proof. The sufficiency follows easily from Theorem 2.12. Conversely suppose $T = \begin{pmatrix} R & I \\ 0 & R \end{pmatrix}$ is VNL. Then by Theorem 2.12, R is regular and for any idempotent e of R , either $eI = 0$ or $(1 - e)I = 0$. Now for any $0 \neq a \in I$, $aR = eR$ for some idempotent $0 \neq e$ in R . As $eI \neq 0$, $(1 - e)I = 0$ and so $I \subseteq eR = aR \subseteq I$. Thus $I = eR = aR$ for every nonzero $a \in I$. This also shows that for any $a \in I$, $aI = I$, implying that $I = eR$ is a simple, commutative, regular ring. Thus I is a field and $R = I \times (1 - e)R$. \square

We now show that the previous result also holds for non-commutative rings that do not have infinite set of orthogonal idempotents.

Corollary 2.17. *Let R be a ring which does not have infinite set of orthogonal idempotents and let I be a non-zero two sided ideal of R . Then $\begin{pmatrix} R & I \\ 0 & R \end{pmatrix}$ is VNL if and only if $R = D \times S$ where D is a division ring, S is a semisimple ring and $I = D \times 0$.*

Proof. Sufficiency is clear from Theorem 2.12. If $T = \begin{pmatrix} R & I \\ 0 & R \end{pmatrix}$ is VNL then R is regular by Lemma 2.4. As R does not have infinite set of orthogonal idempotents, R is semisimple. Thus $R = M_{n_1}(D_1) \times M_{n_2}(D_2) \times \dots \times M_{n_r}(D_r)$, where D_i 's are division rings. So $I = I_1 \times I_2 \times \dots \times I_r$ where each $I_i = 0$ or $M_{n_i}(D_i)$. For any central idempotent $e \in R$, by Theorem 2.12, either $eI = 0$ or $(1 - e)I = 0$. So it is clear that exactly one I_i is non-zero and we may assume I_1 is non-zero. So $T = \begin{pmatrix} M_{n_1}(D_1) \times S & M_{n_1}(D_1) \times 0 \\ 0 & M_{n_1}(D_1) \times S \end{pmatrix}$.

Now $E = \begin{pmatrix} (1, 0) & (0, 0) \\ 0 & (1, 0) \end{pmatrix}$ is a central idempotent in T and so $ET = \begin{pmatrix} M_{n_1}(D_1) & M_{n_1}(D_1) \\ 0 & M_{n_1}(D_1) \end{pmatrix}$ is VNL. So by Corollary 2.13, $n_1 = 1$. \square

We will need the following result, which was proved by Chen and Ying in [4]. As the paper is yet to appear, we give their proof below.

Lemma 2.18. *If R is VNL then so is eRe for any idempotent e in R .*

Proof. Let $a \in eRe$. Suppose a is not regular in R . Then $1 - a$ is regular. Suppose $1 - a = (1 - a)b(1 - a)$, for some $b \in R$. Then $e - a = e(1 - a)e = e(1 - a)b(1 - a)e = (e - a)ebe(e - a)$. \square

3. Characterizations of abelian VNL rings

In this section we characterize abelian VNL rings.

Theorem 3.1. *An abelian ring R is VNL if and only if it is an exchange ring such that for every idempotent e of R , either eRe or $(1 - e)R(1 - e)$ is regular.*

Proof. The ‘only if’ part follows from Lemma 2.4 and the fact that every VNL ring is an exchange ring. Conversely, suppose that R is an abelian

exchange ring such that for every idempotent e of R , either eRe or $(1 - e)R(1 - e)$ is regular. Let $a \in R$, then as R is an exchange ring, there exists an idempotent e such that $e \in aR$ and $1 - e \in (1 - a)R$. So $aR + (1 - e)R = R$ and $eR + (1 - a)R = R$ implying that $eaR = eR$ and $(1 - e)(1 - a)R = (1 - e)R$. Thus ea and $(1 - e)(1 - a)$ are both regular. Now if $eRe = eR$ is regular, then $e(1 - a)$ is regular. So $1 - a = e(1 - a) + (1 - e)(1 - a)$ is regular. Similarly, if $(1 - e)R(1 - e)$ is regular, then a is regular. \square

Recall that a ring R is said to be potent if idempotents lift modulo $J(R)$ and every right ideal not contained in $J(R)$ contains a non-zero idempotent. Every exchange ring is potent and a potent ring without infinite set of orthogonal idempotents is exchange. The following example shows that the above result is not true even for commutative potent rings.

Example 3.2. Let $R = \{(q_1, q_2, \dots, q_n, z, z, z, \dots) : n \geq 1, q_i \in \mathbb{Q}, z \in \mathbb{Z}\}$. It is easy to see that every non-zero ideal of R contains a non-zero idempotent. Also for any idempotent $e \in R$, either eRe or $(1 - e)R(1 - e)$ is regular but R is not VNL as \mathbb{Z} , which is a homomorphic image of R , is not VNL.

In ([3], Lemma 2.7) Chen and Tong have shown that if R is an abelian VNL ring, then $R/M(R)$ is a local ring. The following example shows that this may not be true for non-abelian rings:

Example 3.3. Let $R = T_2(D)$, where D is a division ring. By Corollary 2.13, R is a VNL ring which is not regular. We will show that $M(R) = 0$. Let $e = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$ be a non-zero idempotent in R . It is enough to show that eR is not regular. It is clear that $a, c \in \{0, 1\}$. If $a = c = 1$, then $eR = R$. Also if $a = c = 0$, then as e is in $J(R)$, $b = 0$. If $e = \begin{pmatrix} 1 & b \\ 0 & 0 \end{pmatrix}$, then as $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ is in eR , eR is not a regular right ideal. Lastly if $e = \begin{pmatrix} 0 & b \\ 0 & 1 \end{pmatrix}$, then as $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ is in Re , Re is not a regular left ideal. Thus $M(R) = 0$ and so $R/M(R)$ is not local.

The following lemma will give us another characterization of abelian VNL rings.

Lemma 3.4. *Let I be a regular ideal of a ring R . Then R is VNL if and*

only if R/I is VNL.

Proof. As any factor ring of a VNL ring is clearly VNL, we only have to prove the ‘if’ part. Suppose R/I is VNL and $a \in R$. Then either $a + I$ or $1 - a + I$ is regular in R/I . In particular, either $a - axa \in I$ or $1 - a - (1 - a)y(1 - a) \in I$ for some $x, y \in R$. As I is a regular ideal, either $a - axa$ or $(1 - a) - (1 - a)y(1 - a)$ is a regular element of R . Thus by McCoy’s Lemma, either a or $1 - a$ is regular in R showing that R is VNL. \square

In view of ([3], Lemma 2.7) and above Lemma, the following characterization of abelian VNL rings is immediate.

Theorem 3.5. *Let R be an abelian ring. Then R is VNL if and only if $R/M(R)$ is a local ring.*

4. Characterization of semiperfect VNL rings

In this section we characterize VNL rings without infinite set of orthogonal idempotents, and also the VNL rings R which have a primitive idempotent e such that eRe is not a division ring. As VNL rings without infinite set of orthogonal idempotents are semiperfect, we get a characterization of semiperfect VNL rings.

Note that if e is an idempotent in a ring R , then

$$R \cong \begin{pmatrix} S & X \\ Y & T \end{pmatrix},$$

where $S = eRe$, $T = (1 - e)R(1 - e)$, $X = eR(1 - e)$ is a (S, T) -bimodule and $Y = (1 - e)Re$ is a (T, S) -bimodule such that $XY \subseteq S$ and $YX \subseteq T$. We will be tacitly using this representation of rings below and we will also be using X and Y in place of XE_{12} and YE_{21} .

Lemma 4.1. *If $R = \begin{pmatrix} S & X \\ Y & T \end{pmatrix}$ such that $XY = 0$ or $YX = 0$, then $X, Y \subseteq J(R)$.*

Proof. If $XY = 0$, then it is easy to see that $\begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix}$ is a quasi-regular right ideal of R and $\begin{pmatrix} 0 & 0 \\ Y & 0 \end{pmatrix}$ is a quasi-regular left ideal of R . Similarly if

$YX = 0$ then $\begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix}$ is a quasi-regular left ideal of R and $\begin{pmatrix} 0 & 0 \\ Y & 0 \end{pmatrix}$ is a quasi-regular right ideal of R . \square

Lemma 4.2. *Let e_1 and e_2 be two local idempotents of a ring. Then either $e_1R \cong e_2R$ or $e_1Re_2 \subseteq J(R)$ and $e_2Re_1 \subseteq J(R)$.*

Proof: Suppose $e_1R \not\cong e_2R$. Then for any $r \in R$, $e_1re_2R \neq e_1R$. Because otherwise the map from $e_2R \rightarrow e_1R$ given by the left multiplication with e_1re_2 splits implying that $e_1R \cong e_2R$. Hence e_1re_2R is a proper submodule of e_1R , which has a unique maximal submodule e_1J . Thus $e_1re_2R \subseteq e_1J \subseteq J$ for every r implying that $e_1Re_2 \subseteq J(R)$. Similarly $e_2Re_1 \subseteq J(R)$. \square

Corollary 4.3. *A semiperfect ring R with $1 = e_1 + e_2$, where e_1, e_2 are orthogonal primitive idempotents, is VNL if and only if R is isomorphic to one of the following:*

- (1) $M_2(D)$ for some division ring D .
- (2) $\begin{pmatrix} D & X \\ Y & L \end{pmatrix}$, where D is a division ring, L is a local ring such that $XY = 0$.

In particular, if $J(R) = 0$, then either $R \cong M_2(D)$ or $R \cong \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix}$, where D_i 's and D are division rings.

Proof. If $e_1R \cong e_2R$ then $R \cong M_2(e_1Re_1)$, where e_1Re_1 is a local ring. So by Lemma 2.4, e_1Re_1 is a division ring. If $e_1R \not\cong e_2R$, then by Lemma 4.2, e_1Re_2 and e_2Re_1 are contained in $J(R)$. Again by Lemma 2.4, either e_1Re_1 or e_2Re_2 is a division ring. We may assume that e_1Re_1 is a division ring. Then $e_1Re_2Re_1 \subseteq e_1Re_1 \cap J(R) = e_1Je_1 = 0$. So by taking $D = e_1Re_1$, $L = e_2Re_2$, $X = e_1Re_2$ and $Y = e_2Re_1$, we see that $R \cong \begin{pmatrix} D & X \\ Y & L \end{pmatrix}$ is as in (2) above. \square

Corollary 4.4. *Let R be a semiperfect ring with $1 = e_1 + e_2 + e_3$, where e_1, e_2, e_3 is an orthogonal set of primitive idempotents. Then R is VNL if and only if R is one of the following:*

- (1) $M_3(D)$ for some division ring D .

(2) $\begin{pmatrix} S & X \\ Y & L \end{pmatrix}$, where S is semisimple and L is a local ring with $XY = 0$.

(3) $\begin{pmatrix} T & X \\ Y & D \end{pmatrix}$ where $T \cong \begin{pmatrix} D_1 & X_1 \\ Y_1 & D_2 \end{pmatrix}$ is an NJ ring (see Example 2.1(2) above), D a division ring with $YX = 0$.

Proof. If $e_i R \cong e_j R$ for all i, j , then $R \cong M_3(D)$ for some division ring D .

It is clear that

$$R \cong \begin{pmatrix} (1 - e_1)R(1 - e_1) & (1 - e_1)Re_1 \\ e_1R(1 - e_1) & e_1Re_1 \end{pmatrix}. \quad (A)$$

If e_1Re_1 is local but not a division ring then $(1 - e_1)R(1 - e_1)$ is a semisimple ring implying that e_2Re_2 and e_3Re_3 are division rings. So by Lemma 4.2, $e_1Re_2, e_2Re_1, e_1Re_3, e_3Re_1$ are all contained in $J(R)$ and so $(1 - e_1)Re_1(1 - e_1) = 0$. So R , in view of (A), is as in (2) above.

Now suppose all e_iRe_i are division rings. If $e_2R \cong e_3R$ but $e_1R \not\cong e_2R$, then

$$(1 - e_1)R(1 - e_1) \cong M_2(D)$$

for some division ring D and, by Lemma 4.2,

$$(1 - e_1)Re_1R(1 - e_1) = 0 = e_1R(1 - e_1)Re_1.$$

Thus R as given in (A), is again as in (2) above.

Lastly assume that $e_1R \not\cong e_2R \not\cong e_3R$. Then

$$(1 - e_1)R(1 - e_1) \cong \begin{pmatrix} D_1 & X_1 \\ Y_1 & D_2 \end{pmatrix},$$

with $X_1Y_1 = 0 = Y_1X_1$, where $D_1 = e_2Re_2$ and $D_2 = e_3Re_3$. In view of Lemma 4.2, it is clear that $e_1R(1 - e_1)Re_1 = 0$. This in view of (A) implies that R is as in (3) above. \square

Lemma 4.5. (1) Let $R = \begin{pmatrix} S & X \\ Y & L \end{pmatrix}$, where L is a local and S is a regular ring such that $XY = 0$. If $a = \begin{pmatrix} s & x \\ y & u \end{pmatrix} \in R$ such that $u \in L$ is a unit, then a is regular. In particular, R is a VNL ring.

(2) Let $S = \begin{pmatrix} T & X \\ Y & D \end{pmatrix}$, where $T \cong \begin{pmatrix} D_1 & X_1 \\ Y_1 & D_2 \end{pmatrix}$ is an NJ ring, D a division

ring with $YX = 0$. If $b = \begin{pmatrix} t & x \\ y & d \end{pmatrix} \in S$, then b is regular under any of the following conditions:

- (a) If t is regular in T and $d \neq 0$.
- (b) If t is a unit in T .

Proof. Suppose $a = \begin{pmatrix} s & x \\ y & u \end{pmatrix} \in R$ where $u \in L$ is a unit. Then clearly

$$a = \begin{pmatrix} s & xu^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ y & u \end{pmatrix}$$

As $\begin{pmatrix} 1 & 0 \\ y & u \end{pmatrix}$ is a unit in R and $\begin{pmatrix} s & xu^{-1} \\ 0 & 1 \end{pmatrix}$ is regular by Proposition 2.8, a is clearly regular.

Now suppose

$$b = \begin{pmatrix} t & x \\ y & d \end{pmatrix} \in S,$$

where t is regular in T and $d \neq 0$. As all the regular elements of T are unit regular, $t = eu$ for some idempotent e and unit u in T . Then $c = \begin{pmatrix} u & x \\ y & d \end{pmatrix}$ is a unit in S with

$$c^{-1} = \begin{pmatrix} (u - xd^{-1}y)^{-1} & -u^{-1}xd^{-1} \\ -d^{-1}y(u - xd^{-1}y)^{-1} & d^{-1} \end{pmatrix}.$$

So b is regular if and only if bc^{-1} is regular. Now

$$bc^{-1} = \begin{pmatrix} t_1 & -exd^{-1} + xd^{-1} \\ 0 & 1 \end{pmatrix},$$

where $t_1 = t(u - xd^{-1}y)^{-1} - xd^{-1}y(u - xd^{-1}y)^{-1}$. By Lemma 4.1, $x \in J(S)$ implying that $xd^{-1}y(u - xd^{-1}y)^{-1} \in J(T)$. As $t(u - xd^{-1}y)^{-1}$ is regular in T and only non-regular elements of T are the elements of $J(T)$, t_1 is regular. So by Proposition 2.8, bc^{-1} is regular.

Lastly if t is a unit and $d = 0$, then again b is regular with von Neumann inverse as $\begin{pmatrix} t^{-1} & 0 \\ 0 & 0 \end{pmatrix}$. \square

We are now ready to characterize semiperfect VNL rings.

Theorem 4.6. *A semiperfect ring R is VNL if and only if $R = A \times B$, where A is a semisimple ring and B is one of the following:*

(1) *Semisimple.*

(2) $R_1 = \begin{pmatrix} S & X \\ Y & L \end{pmatrix}$, where L is a local ring, S is a semisimple ring such that $XY = 0$.

(3) $R_2 = \begin{pmatrix} T & X \\ Y & D \end{pmatrix}$, where $T \cong \begin{pmatrix} D_1 & X_1 \\ Y_1 & D_2 \end{pmatrix}$ is an NJ ring, D a division ring such that $YX = 0$ (clearly this case occurs in semiperfect rings with $1 = e_1 + e_2 + e_3$ only).

Proof. In view of Lemma 4.5(1), R_1 is VNL. Let $a = \begin{pmatrix} t & x \\ y & d \end{pmatrix} \in R_2$. If t is regular in T and $d \neq 0$, then a is regular by Lemma 4.5(2). Also by Lemma 4.5(2), a is regular if t is a unit in T . Now assume that t is not a unit in T . As non-regular elements of T are in $J(T)$, $1 - t$ is regular in T . If $d \neq 1$, then $1 - a = \begin{pmatrix} 1 - t & -x \\ -y & 1 - d \end{pmatrix}$ which is regular by Lemma 4.5(2). Now suppose that $d = 1$. Then if t is regular in T , then a is regular. If t is not regular, then $t \in J(T)$ and so $1 - t$ is a unit in T . Then $1 - a = \begin{pmatrix} 1 - t & -x \\ -y & 0 \end{pmatrix}$ is regular by Lemma 4.5 (2). Thus if $R \cong A \times B$, with A semisimple and B either semisimple or isomorphic to R_1 or R_2 , then R is VNL.

Conversely, let R be a semiperfect VNL ring. In view of the block decomposition of semiperfect rings and Lemma 2.4, $R \cong A \times B$, where A is semisimple and B is a semiperfect VNL ring with no non-trivial central idempotents. So we assume without loss of generality that R is a semiperfect VNL ring without any non-trivial central idempotent. In the proof below, we will call an idempotent e_i , $1 \leq i \leq n$, single if $e_i R \not\cong e_j R$ for any $j \neq i$. Let $1 = e_1 + e_2 + \dots + e_n$ where e_i 's are orthogonal primitive idempotents. We have already discussed the case $n \leq 3$ in Corollary 4.3 and Corollary 4.4. We assume that $n \geq 4$. Clearly for any i ,

$$R \cong \begin{pmatrix} (1 - e_i)R(1 - e_i) & (1 - e_i)Re_i \\ e_i R(1 - e_i) & e_i Re_i \end{pmatrix} \quad (B)$$

Suppose first that there exist e_i such that $e_i Re_i$ is local but not a division ring. Then by Lemma 2.4, $(1 - e_i)R(1 - e_i)$ is a semisimple ring. In particular, each

e_jRe_j is a division ring whenever $j \neq i$. Thus $e_iR \not\cong e_jR$ and so by Lemma 4.2, e_iRe_j, e_jRe_i are contained in $J(R)$ for $j \neq i$. But as $(1 - e_i)R(1 - e_i)$ is semisimple, $(1 - e_i)Re_iR(1 - e_i) = 0$. So R , in view of (B), is isomorphic to R_1 in this case.

Now assume that each e_iRe_i is a division ring. Note that if

$$\begin{pmatrix} M_{n_1}(D_1) & X \\ Y & M_{n_2}(D_2) \end{pmatrix} \text{ with } XY = 0 = YX$$

is VNL, then either it is semisimple or one of n_1 or n_2 is equal to 1. So if each $e_iR \cong e_jR$ for some $j \neq i$ then R is a semisimple ring. Now assume there exist e_i such that $e_iR \not\cong e_jR$ for any j . If for each e_j , $i \neq j$, there exist e_k with $k \neq j$ such that $e_jR \cong e_kR$, then as mentioned above, $(1 - e_i)R(1 - e_i)$ is a semisimple ring and by Lemma 4.2, $(1 - e_i)Re_iR(1 - e_i) = 0 = e_iR(1 - e_i)Re_i$. So R , in view of (B), is isomorphic to R_1 in this case also, with L a division ring.

So we assume that there are more than one single idempotents say e_1, e_2, \dots, e_r . If f_{r+1}, \dots, f_m denote the sum of isomorphic e_i 's. Then

$$1 = e_1 + \dots + e_r + f_{r+1} + \dots + f_m$$

Let $e = f_{r+1} + \dots + f_m$, by Lemma 2.4, eRe is clearly semisimple. Suppose $(1 - e)R(1 - e)$ is also semisimple. If $(e_i + f_j)R(e_i + f_j)$ is regular for every i, j , then R is semisimple. We now assume that $(e_i + f_j)R(e_i + f_j)$ is not regular for some i, j . If $f_j = e_{i_1} + e_{i_2} + \dots$ then as $(e_i + e_{i_1})R(e_i + e_{i_1})$ is also not regular but $(e_i + e_{i_1} + e_k + e_{i_2})R(e_i + e_{i_1} + e_k + e_{i_2})$ is VNL for each $k \neq i$ (see Lemma 2.18), so by Lemma 2.4, $(e_k + e_{i_2})R(e_k + e_{i_2})$ is regular for each $k \neq i$ and hence $(e_k + f_j)R(e_k + f_j)$ is regular for each $k \neq i$. So $e_kRf_j = 0 = f_jRe_k$ for each $k \neq i$. Thus $(1 - e_i)R(1 - e_i)$ is semisimple, this with Lemma 4.2 implies

$$(1 - e_i)Re_i(1 - e_i) = 0 = e_iR(1 - e_i)Re_i$$

So R , in view of (B), is isomorphic to R_1 , with L a division ring.

Lastly we assume that $(1 - e)R(1 - e)$ is not semisimple. So there exist e_i, e_j such that $(e_i + e_j)R(e_i + e_j)$ is not semisimple and therefore by Lemma 2.4, $(1 - (e_i + e_j))R(1 - (e_i + e_j))$ is semisimple. As $n \geq 4$, we can pick k, l not equal to i or j . Then by Lemma 2.18 and Lemma 2.4, either $(e_k + e_i)R(e_k + e_i)$ or $(e_l + e_j)R(e_l + e_j)$ is semisimple. Assume that $(e_k + e_i)R(e_k + e_i)$ is

semisimple. If $(e_k + e_j)R(e_k + e_j)$ is also semisimple and e_k is single then it is clearly central. If e_k is not single, then the corresponding f_s is central. So assume that $(e_k + e_j)R(e_k + e_j)$ is not semisimple. Then for any t not in $\{i, j, k\}$, $(e_t + e_i)R(e_t + e_i)$ is semisimple by Lemma 2.4 and Lemma 2.18. In particular, $(1 - e_j)R(1 - e_j)$ is semisimple. So by Lemma 4.2, $(1 - e_j)Re_jR(1 - e_j) = 0 = e_jR(1 - e_j)Re_j$. So using (B) with $i = j$, we have that R is isomorphic to R_1 , with L a division ring. \square

A ring R is called right n-VNL-ring if $a_1R + a_2R + \dots + a_nR = R$ implies that some a_i is regular for some i . In [4], it was shown that the semiperfect ring VNL ring $T_3(D)$ is not 3-VNL. We prove that every semiperfect VNL ring is 2-VNL.

Theorem 4.7. *A semiperfect VNL ring R is 2-VNL.*

Proof. We will use the characterization of semiperfect VNL rings as given in Theorem 4.6. We first show that R_1 is 2-VNL. Suppose $A = \begin{pmatrix} s_1 & x_1 \\ y_1 & l_1 \end{pmatrix}$ and $B = \begin{pmatrix} s_2 & x_2 \\ y_2 & l_2 \end{pmatrix}$ are elements of R_1 such that $AR_1 + BR_1 = R_1$. So there exist elements $C = \begin{pmatrix} s_3 & x_3 \\ y_3 & l_3 \end{pmatrix}$ and $D = \begin{pmatrix} s_4 & x_4 \\ y_4 & l_4 \end{pmatrix}$ in R_1 such that $AC + BD = 1$ implying that

$$\begin{pmatrix} s_1s_3 + s_2s_4 & s_1x_3 + x_1l_3 + s_2x_4 + x_2l_4 \\ y_1s_3 + l_1y_3 + y_2s_4 + l_2y_4 & y_1x_3 + l_1l_3 + y_2x_4 + l_2l_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

As $XY = 0$ in R_1 , by Lemma 4.1, $X, Y \subseteq J(R_1)$. Thus

$$\begin{pmatrix} 0 & 0 \\ 0 & y_1x_3 + y_2x_4 \end{pmatrix} \in J(R_1)$$

and so $\begin{pmatrix} 1 & 0 \\ 0 & 1 - y_1x_3 - y_2x_4 \end{pmatrix}$ is a unit in R_1 implying that $\begin{pmatrix} 1 & 0 \\ 0 & l_1l_3 + l_2l_4 \end{pmatrix}$ is a unit in R_1 . So $l_1l_3 + l_2l_4$ is a unit in L . As L is a local ring, either l_1 or l_2 is a unit in L . So in view of Lemma 4.5, either A or B is regular. Thus R_1 is 2-VNL. We now show that R_2 is 2-VNL. Suppose $P = \begin{pmatrix} t_1 & x_1 \\ y_1 & d_1 \end{pmatrix}$ and $Q = \begin{pmatrix} t_2 & x_2 \\ y_2 & d_2 \end{pmatrix}$ are elements of R_2 such that $PR_2 + QR_2 = R_2$. If t_1

or t_2 is a unit in T , then in view of Lemma 4.5, the corresponding element P or Q is regular in R_2 . So suppose neither t_1 and nor t_2 is unit in T . As $PR_2 + QR_2 = R_2$, there exist $U = \begin{pmatrix} t_3 & x_3 \\ y_3 & d_3 \end{pmatrix}$ and $V = \begin{pmatrix} t_4 & x_4 \\ y_4 & d_4 \end{pmatrix}$ in R_2 such that $PU + QV = 1$, implying that

$$\begin{pmatrix} t_1t_3 + t_2t_4 + x_1y_3 + x_2y_4 & t_1x_3 + x_1d_3 + t_2x_4 + x_2d_4 \\ y_1t_3 + d_1y_3 + y_2t_4 + d_2y_4 & d_1d_3 + d_2d_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

So $d_1d_3 + d_2d_4 = 1$, implying that at least one of d_1 and d_2 is a unit. Again as $YX = 0$ in R_2 , by Lemma 4.1, $X, Y \subseteq J(R_2)$ and so $\begin{pmatrix} x_1y_3 + x_2y_4 & 0 \\ 0 & 0 \end{pmatrix} \in J(R_2)$, $\begin{pmatrix} 1 - x_1y_3 - x_2y_4 & 0 \\ 0 & 1 \end{pmatrix}$ is a unit in R_2 . So $t_1t_3 + t_2t_4$ is a unit in T . As T is an NJ ring and none of t_1 and t_2 is a unit, it is easy to see that both t_1 and t_2 are regular in T . Also since one of d_1 and d_2 is a unit in D , so by Lemma 4.5, the corresponding element P or Q is regular in R_2 . \square

We now characterize VNL rings in which there is a primitive idempotent e such that eRe is not a division ring.

Theorem 4.8. *Let R be a ring with a primitive idempotent e such that eRe is not a division ring. Then R is VNL if and only if $R \cong \begin{pmatrix} S & X \\ Y & L \end{pmatrix}$, where L is a local ring, S is a regular ring and $XY = 0$.*

Proof. The ‘if’ part follows from Lemma 4.5(1). Now suppose that R is VNL and $e \in R$ is a primitive idempotent such that eRe is not a division ring. So eRe is a local ring and, by Lemma 2.4, $(1 - e)R(1 - e)$ is a regular ring. We have

$$R \cong \begin{pmatrix} (1 - e)R(1 - e) & (1 - e)Re \\ eR(1 - e) & eRe \end{pmatrix}.$$

We now show that $eR(1 - e) \subseteq J(R)$. Note that if for any element $r \in R$, $er(1 - e)R = eR$, then eR will be isomorphic to a summand of $(1 - e)R$. But as corner rings of regular rings are regular, eRe is regular and hence a division ring, a contradiction. So $er(1 - e)R$ is a proper submodule of a local module eR implying that $er(1 - e)R \subseteq eJ(R)$. So $eR(1 - e) \subseteq J(R)$. In particular, $(1 - e)ReR(1 - e) \in J(R) \cap (1 - e)R(1 - e) = 0$, as $(1 - e)R(1 - e)$ is a regular ring. \square

5. A Sufficient Condition

A ring R is called semipotent if every right ideal not contained in $J(R)$ contains a nonzero idempotent. In general we have

$$\text{NJ} \implies \text{VNL} \implies \text{Exchange} \implies \text{Potent} \implies \text{Semipotent},$$

with none of the implications reversible. We give below a sufficient condition for all these classes of rings to coincide.

Lemma 5.1. *Let R be a semipotent ring without central idempotents such that $J(R) \neq 0$ but $J(eRe) = 0$ for every proper idempotent e of R . Then the following hold:*

(1) *For every proper idempotent e of R , $eR(1-e)$ and $(1-e)Re$ are contained in $J(R)$.*

(2) *If $0 \neq e = e^2$ is such that $ae = a = ea$ for every a in $J(R)$, then e is central and hence $e = 1$*

Proof. First note that if a is in $J(R)$, then for every proper idempotent e of R , $a = ea + ae$ as $(1-e)J(1-e) = 0$ and $eJe = 0$. Let e be proper idempotent of R . Now $(1-e)JeR(1-e) = 0$ as it is contained in $(1-e)J(1-e)$, so $JeR(1-e) = eJeR(1-e) = 0$. Also $eR(1-e)J(1-e) = 0$ implying that $eR(1-e)J = eR(1-e)Je = 0$. Thus $eR(1-e)J = JeR(1-e) = 0$ and so $eR(1-e) \subseteq \text{Ann}(J(R))$. If $eR(1-e) \not\subseteq J(R)$ then $\text{Ann}J(R) \not\subseteq J(R)$ and as R is semipotent, there exist $0 \neq f = f^2 \in \text{Ann}J(R)$. Then as $J(R) \neq 0$, $f \neq 1$ and hence f is proper. So $a = af + fa = 0$ for every a in $J(R)$ implying $J(R) = 0$, a contradiction. Hence $eR(1-e) \subseteq J(R)$. Similarly $(1-e)Re \subseteq J(R)$.

Now suppose $0 \neq e = e^2$ is such that $ae = ea = a$ for every a in $J(R)$. If $e = 1$, then nothing to prove. If e is proper, then as $eR(1-e)$ and $(1-e)Re$ are contained in $J(R)$, $e.er(1-e) = er(1-e)e = 0$ for every r in R implying that $er(1-e) = 0$, similarly $(1-e)re = 0$ for all r in R and so e is central. \square

Proposition 5.2. *Let R be a ring without central idempotents such that $J(R) \neq 0$ but $J(eRe) = 0$ for every proper idempotent e of R . Then the following are equivalent:*

- (1) *R is a VNL ring.*
- (2) *R is an exchange ring.*

- (3) R is a potent ring.
- (4) R is a semipotent ring.
- (5) R is an NJ ring.

Proof. The implications $(1) \implies (2) \implies (3) \implies (4)$ and $(5) \implies (1)$ hold in general. So we only have to prove the implication $(4) \implies (5)$.

Note that if a is in $J(R)$, then for every proper idempotent e of R , $a = ea + ae$. Now we prove that if e is a proper idempotent of R then eRe has only trivial idempotents. Suppose f is a proper idempotent in eRe , then $ef = fe = f$. Clearly $e - f \neq 0$, also it is easy to see that $e - f \neq 1$ and hence $e - f$ is a proper idempotent of R . Now for any $a \in J(R)$, $a = af + fa$. Also $af = af(e - f) + (e - f)af = eaf$ and $fa = fa(e - f) + (e - f)fa = fae$. Then as $a = af + fa$, $ea = eaf + efa = af + fa = a$ and $ae = afe + fae = af + fa = a$. So we have $ea = ae = a$ for every a in $J(R)$ and thus by Lemma 5.1, $e = 1$, a contradiction. Thus eRe has only two idempotents. Now as R semipotent implies eRe semipotent, eRe is a local ring for every proper idempotent e of R . Also as $J(eRe) = 0$, eRe is a division ring.

Now if R has no proper idempotent then R is local and hence NJ. If e is a proper idempotent in R , then eRe and $(1 - e)R(1 - e)$ are division rings and in view of Lemma 5.1, $eR(1 - e)Re = 0 = (1 - e)ReR(1 - e)$. Thus

$$R \cong \begin{pmatrix} D_1 & X \\ Y & D_2 \end{pmatrix},$$

where D_1, D_2 are division rings and $XY = 0 = YX$ and hence by Nicholson's characterization of NJ rings, R is an NJ ring (see [6]). \square

In Theorem 3.1, we proved that an abelian ring R is VNL if and only if it is an exchange ring with the property that for every idempotent e of R , one of the two corner rings eRe or $(1 - e)R(1 - e)$ is regular. So one may ask:

Question 5.3. *Let R be an arbitrary exchange ring with the property that for every idempotent e of R , one of eRe and $(1 - e)R(1 - e)$ is regular. Then is R a VNL ring?*

In Theorem 4.7, we proved that a semiperfect VNL ring is 2-VNL. So the following natural question arises:

Question 5.4 *Is every VNL ring 2-VNL?*

References

- [1] B. Brown and N. McCoy, *The Maximal Regular Ideal of a Ring*, Proc. Amer. Math. Soc. **1**(1950), 165-171.
- [2] V. P. Camillo, H. P. Yu , *Exchange Rings, Units and Idempotents*, Comm. Algebra **22**(1994), 4737-4749.
- [3] W. X. Chen and W. T. Tong, *On Noncommutative VNL-rings and GVNL-rings*, Glasgow Math. J., **48**(2006), 11-17.
- [4] J. Chen and Z. Ying, *On VNL-rings and n -VNL-rings*, to appear.
- [5] M. Contessa, *On Certain Classes Of PM-rings*, Communications in Algebra, **12**(1984), 1447-1469.
- [6] W. K. Nicholson, *Rings whose elements are Quasi-Regular or Regular*, Aequationes Mathematicae **9**(1973), 64-70.
- [7] W. K. Nicholson, *Lifting Idempotents and Exchange Rings*, Trans. Amer. Math. Soc. **229**(1977), 269-278.
- [8] E. A. Osba, M. Henriksen and O. Alkam, *Combining Local and von Neumann regular rings*, Communications in Algebra, **32**(2004), 2639-2653.
- [9] E. A. Osba, M. Henriksen, O. Alkam and F. A. Smith, *The Maximal Regular Ideal of Some Commutative Rings*, Comment. Math. Univ. Carolinae, **47**(2006), 1-10.